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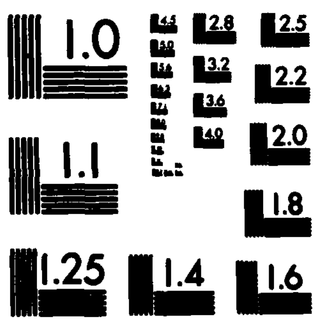

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Stability analyses are performed on a general type of vortex flows with density inhomogeneity and with both the axial and azimuthal magnetic fields. A sufficient condition for stability, subject to arbitrary perturbations, is derived. The condition, however, involves the complex eigenfrequency. Nevertheless, we are able to examine how the flow profile influences the flow characteristics by studying three general types of perturbations. Analytical solutions are obtained to support the resultant arguments.

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20. ABSTRACT (Continued)

Unlike the axial velocity which always destabilizes the flow, the azimuthal velocity can convey either stabilizing or destabilizing effects depending on the gradient it possesses and the centrifugal force field it creates. This characteristic is shown in all the three general types of perturbations to the flow.

For flows subject to azimuthal perturbations, an upper bound on the complex phase velocity, which is reminiscent of the semi-ellipse theorem in the non-hydromagnetic case, is found for some flow profiles satisfying a constraint. Even though it cannot be observed directly from the sufficiency condition or from the bound on the unstable waves, we are able to show that the presence of the magnetic field, regardless of the detailed profile, always stabilizes azimuthal perturbations. Such an argument is also supported by exact solutions to the governing stability equations. For uniform rotation and constant angular Alfvén velocities, azimuthal instabilities can only occur when negative density gardients exist within the flow domain. Furthermore, the phase velocity for such instabilities must lie on a semi-circle in the complex phase velocity plane independent of the detailed density profile. Both the semi-ellipse and semi-circle bounds clearly demonstrate that the unstable waves do not necessarily propagate against the basic rotation. In other words, the westward drift phenomenon earlier derived for flows subject to arbitrary distrubances cannot in general be applied to azimuthal modes.

CONTENTS

ON THE STABILITY OF VORTEX MOTIONS IN THE PRESENCE OF MAGNETIC FIELDS

INTRODUCTION

In an important paper on the hydrodynamic and hydromagnetic stability of nondissipative swirling flows, Howard & Gupta (1962) presented many interesting aspects on stability characteristics of vortex motion with and without axial velocity. Even though they restricted themselves mostly to homogeneous fluids subject to axisymmetric perturbations, their generalization of the Richardson criterion and of the semi-circle theorem to axisymmetric steady flows presented a relatively simple insight into the type of parallel flows complicated by the cylindrical geometry. Acheson (1972, 1973) later extended the analyses into the same type of flows subject to non-axisymmetric perturbations. A review paper on this subject was given by Acheson & Hide (1973).

By considering a homogeneous fluid rotating uniformly in a radius-dependent magnetic field, Acheson (1972) derived some sufficient conditions for stability of the flow. The difference between the axisymmetric and nonaxisymmetric modes were brought out by assuming the wavelengths in the radial direction to be small compared with the radius. In addition, the stability phenomenon that all non-axisymmetric unstable waves must propagate against the basic rotation, i.e., the westward drift, was proved to prevail with a restriction on the axial and azimuthal components of the magnetic field. With the help of the Boussinesq approximation, the author later (Acheson 1973) generalized the westward drift phenomenon to heterogeneous fluids rotating differentially. Also based on this westward drift, a quadrant theorem reminiscent of the semi-circle theorem encountered in two-dimensional stratified flows was derived for slow amplifying waves.

It should be emphasized, as also noted explicitly by Acheson (1973), that the results obtained in his paper are restricted to perturbations with non-zero axial wave numbers. Any attempt to infer that

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the results can be applied to perturbations with zero axial wave numbers (azimuthal modes) may lead to incorrect conclusions on both the hydrodynamic and hydromagnetic problems. This argument will be demonstrated by examining the azimuthal modes in this paper.

In the present investigation, stability analyses are performed on a general type of vortex flow with varying density in the presence of an axial and an azimuthal magnetic field. A sufficiency condition for stability is derived and compared with the results obtained from other methods. The dual role played by the rotating velocity on the stability of vortex motions is separated and revealed by proper transformations. While the shear effect conveyed by the velocity gradient at the shear layer always destabilizes the flow, the centrifugal force generated by the fluid rotation stabilizes or destabilizes the flow depending on whether or not the density increases radially outwards.

For vortex motions subject to azimuthal disturbances, a magnetic field always has a stabilizing effect regardless of its detailed distribution. As an extension of the semi-ellipse theorem in stratified rotating flows (Fung 1982), an upper bound on azimuthal amplifying waves is derived with a restriction. The restriction suggests that the semi-circle in the complex velocity plane does not provide a bound on all the growing azimuthal modes. Furthermore, the upper bound demonstrates that unstable waves do not necessarily drift westward for the hydromagnetic or non-hydromagnetic case. For uniformly rotating flows with constant angular Alfvén velocities, all azimuthal unstable waves must lie on a semi-circle independent of the density distributions. Exact solutions for some special flow profiles are obtained to support the derived stability criteria.

MATHEMATICAL DERIVATION

Consider a vortex flow with a velocity $\bar{U}$ and a magnetic field $\bar{H}$ to be confined within the annular region (r, θ, z) between two rigid, infinite and coaxial cylinders. The fluid having an inhomogeneous density ρ^* is assumed to be inviscid, incompressible, and non-heat-conducting. When gravitational forces and dissipation effects due to viscosity, magnetic resistivity, and thermal diffusivity are neglected, the governing equations for the flow are

$$\rho^* \frac{D\vec{U}}{Dt} = \nabla Q + \frac{\mu}{4\pi} (\vec{H} \cdot \nabla) \vec{H} \tag{1}$$

$$\nabla \cdot \vec{U} = 0 \tag{2}$$

$$\frac{\partial \vec{H}}{\partial t} = \nabla \times (\vec{U} \times \vec{H}) \tag{3}$$

$$\nabla \cdot \vec{H} = 0 \tag{4}$$

$$\frac{D\rho^*}{Dt} = 0 \tag{5}$$

where μ denotes the magnetic permeability. The total pressure Q (including the magnetic pressure) is related to the hydrodynamic pressure P as follows:

$$Q = P + \frac{\mu}{4\pi} |\vec{H}|^2. \tag{6}$$

The boundary conditions for the system governed by Eqs. (1) to (5) are those of perfectly conducting rigid walls.

The flow to be considered has a steady-state, radius-dependent profile as follows: $\Omega(r)$ is the angular velocity, $W(r)$ the axial velocity, $H_\theta(r)$ the azimuthal magnetic field, $H_z(r)$ the axial magnetic field, $Q_0(r)$ the total pressure, and $\rho_0(r)$ the density. Let the flow be perturbed as follows:

$$\begin{aligned}
&\vec{U} = \vec{U}\ [\hat{u},\ r\Omega(r) + \hat{v},\ W(r) + \hat{w}],\\
&\vec{H} = \vec{H}\ [\hat{h}_r,\ H_\theta(r) + \hat{h}_\theta,\ H_z(r) + \hat{h}_z],\\
&Q = Q_0(r) + \hat{q},\\
&\rho^* = \rho_0(r) + \hat{\rho}.
\end{aligned} \tag{7}$$

We further introduce the periodic solutions

$$\hat{\phi} = \phi(r)\ Exp\ [i(kz + m\theta - \omega t)] \tag{8}$$

such that the azimuthal wave number m is an integer, the axial wave number k is real and positive, and the circular frequency $\omega = \omega_r + i\omega_i$ is complex. Within the framework of the normal mode method, the linearized equations for the flow described by Eqs. (1) to (5), subject to small perturbations, are given as follows:

$$\rho_0\ [iNu - 2\Omega v] - \frac{\mu}{4\pi}\ [iN_a h_r - \frac{2H_\theta}{r}\ h_\theta] - r\Omega^2\rho = -Dq \tag{9}$$

$$\rho_0\ [iNv + D^*(r\Omega)u] - \frac{\mu}{4\pi}\ [iN_a h_\theta + D^*(H_\theta)h_r] = -i\ \frac{m}{r}\ q \tag{10}$$

$$\rho_0\ [iNw + (DW)\ u] - \frac{\mu}{4\pi}\ [iN_a h_z + (DH_z)h_r] = -ikq \tag{11}$$

$$D^* u + i\left(\frac{m}{r}\,\mathrm{v} + kw\right) = 0 \tag{12}$$

$$Nh_r - N_a u = 0 \tag{13}$$

$$iNh_\theta - D_*\,(r\Omega)\,h_r - [iN_a\mathrm{v} - D_*\,(H_\theta)u] = -H_\theta\left[D^* u + i\left(\frac{m}{r}\mathrm{v} + kw\right)\right] \tag{14}$$

$$iNh_z - (DW)\,h_r - [iN_a w - (DH_z)\,u] = -H_z\left[D^* u + i\left(\frac{m}{r}\mathrm{v} + kw\right)\right] \tag{15}$$

$$D^* h_r + i\left(\frac{m}{r}\,h_\theta + kh_z\right) = 0 \tag{16}$$

$$iN\rho + (D\rho_0)\,u = 0 \tag{17}$$

where $N = kW + m\Omega - \omega$ is the Doppler-shifted frequency, $N_a = kH_z + m\,\dfrac{H_\theta}{r}$, $D = \dfrac{d}{dr}$, $D^* = D + \dfrac{1}{r}$ and $D_* = D - \dfrac{1}{r}$. The characteristic that Eqs. (3) and (4) represent only three independent partial differential equations, under the present assumption, is reflected in Eqs. (13) to (16).

If we define the angular and axial Alfvén velocities as

$$\Omega_A = \sqrt{\frac{\mu}{4\pi\rho_0}}\,\frac{H_\theta}{r}, \quad W_A = \sqrt{\frac{\mu}{4\pi\rho_0}}\,H_z$$

and the Alfvén frequency as

$$N_A = k\,W_A + m\,\Omega_A,$$

Eqs. (9) to (17) can be combined into two independent differential equations as follows:

$$\left(1 - \frac{N_A^2}{N^2}\right) D^*\left(\frac{u}{N}\right) - \frac{2m\Omega}{Nr}\left(1 - \frac{N_A}{N}\,\frac{\Omega_A}{\Omega}\right)\left(\frac{u}{N}\right) - \frac{i}{\rho_0}\,\frac{k^2 + \dfrac{m^2}{r^2}}{N^2}\,q = 0 \tag{18}$$

$$\left\{\left(1 - \frac{N_A^2}{N^2}\right)\left[(N^2 - \Phi) - (N_A^2 - \Psi_A)\right] - 4\Omega^2\,\frac{N_A^2}{N^2}\left[\left(1 - \frac{\Omega_A}{\Omega}\right)^2 + 2\,\frac{\Omega_A}{\Omega}\left(1 - \frac{N}{N_A}\right)\right]\right\}\left(\frac{u}{N}\right)$$

$$-\frac{i}{\rho_0}\left[\left(1 - \frac{N_A^2}{N^2}\right) Dq + \frac{2m\Omega}{Nr}\left(1 - \frac{\Omega_A}{\Omega}\,\frac{N_A}{N}\right) q\right] = 0 \tag{19}$$

where the Rayleigh-Synge discriminant and the Alfvén discriminant are respectively defined as

$$\Phi = \frac{D\,[\rho_0(r^2\Omega)^2]}{\rho_0 r^3}$$

$$\Psi_A = \frac{r}{\rho_0}\,D\,[\rho_0\Omega_A^2].$$

The equation governing stability of the flow can be obtained by combining Eqs. (18) and (19) as follows:

$$D\,[\rho_0 E(N^2 - N_A^2)D^*\psi] - \left\{D_*\left[\frac{2m\rho_0 E}{r}(N\Omega - N_A\Omega_A)\right]\right.$$
$$\left. + \rho_0\left[N^2 - N_A^2 - \Phi + \Psi_A + 4\Omega^2 - 4k^2E\,\frac{(N\Omega - N_A\Omega_A)^2}{N^2 - N_A^2}\right]\right\}\psi = 0 \qquad (20)$$

where

$$\psi = \frac{u}{N} \qquad (21)$$

and

$$E = \frac{r^2}{m^2 + k^2r^2}.$$

The boundary conditions for the above equation are

$$\psi(r_1) = \psi(r_2) = 0 \qquad (22)$$

where r_1 and r_2 are locations of the solid boundaries. If discontinuities in the flow profile exist within the flow domain, the matching conditions obtained by integrating Eqs. (18) and (19) across the discontinuity surface are

$$<\frac{u}{N}) = 0 \qquad (23)$$

$$<q> - i\left(\frac{u}{N}\right) <\rho_0 r(\Omega^2 - \Omega_A^2)> = 0 \qquad (24)$$

where $<\phi> = \phi_{(R+0)} - \phi_{(R-0)}$ represents the jump condition across the discontinuity surface at $r = R$.

STABILITY CRITERIA

We will investigate the general stability characteristics described by Eq. (20). Due to the complexity of the presence of all the four components in the velocity and magnetic fields, general stability criteria only in terms of the flow profiles are difficult to derive. However, stability conditions for some particular cases can still be observed and discussed in the following.

(1) Sufficient Condition for Stability

Let

$$\psi = N^{-1/2}\phi \tag{25}$$

Equation (20) is then transformed to

$$D\left[\rho_0 E\left(N - \frac{N_A^2}{N}\right) D^*\phi\right] - rD\left\{\frac{\rho_0 E}{r^2}\left[\frac{rDN}{2}\left(1 - \frac{N_A^2}{N^2}\right) + \frac{2m}{N}(N\Omega - N_A\Omega_A)\right]\right\}\phi$$
$$+ \left\{-\frac{1}{4}\frac{\rho_0 E(DN)^2}{N}\left[1 - \frac{N_A^2}{N^2}\right] - \frac{2m\rho_0 EDN}{rN^2}(N\Omega - N_A\Omega_A)\right.$$
$$\left. - \frac{\rho_0}{N}\left[N^2 - N_A^2 - \Phi + \Psi_A + 4\Omega^2 - 4k^2E\frac{(N\Omega - N_A\Omega_A)^2}{N^2 - N_A^2}\right]\right\}\phi = 0 \tag{26}$$

Multiplying the above equation by $r\bar{\phi}$, where the quantity with a bar stands for the complex conjugate, and integrating the resultant equation over the flow domain, we obtain

$$\int \rho_0\left(N - \frac{N_A^2}{N}\right)(E|D^*\phi|^2 + |\phi|^2)rdr$$
$$- \int \frac{\rho_0 E}{r^2}\left[\frac{rDN}{2}\left(1 - \frac{N_A^2}{N^2}\right) + \frac{2m}{N}(N\Omega - N_A\Omega_A)\right] D(r^2|\phi|^2)dr$$
$$+ \int\left\{\frac{1}{4}\frac{\rho_0 E(DN)^2}{N}\left(1 - \frac{N_A^2}{N^2}\right) + \frac{2m\rho_0 EDN}{rN^2}(N\Omega - N_A\Omega_A)\right.$$
$$\left. - \frac{\rho_0}{N}\left[\Phi - \Psi_A - 4\Omega^2 + 4k^2E\frac{(N\Omega - N_A\Omega_A)^2}{N^2 - N_A^2}\right]\right\}|\phi|^2 rdr = 0 \tag{27}$$

with its imaginary part equal to

$$\omega_i\left\{\int \rho_0(E|D^*\phi|^2 + |\phi|^2)rdr + \int\frac{\rho_0 E}{|N|^2}\left\{N_A D|\phi|\right.\right.$$
$$\left. + \left[\frac{N_A}{r} - \frac{2m\Omega_A}{r} - \frac{(kW + m\Omega - \omega_r)N_A DN}{|N|^2}\right]|\phi|\right\}^2 rdr$$
$$+ \int 2k^2\rho_0 E\left[\frac{(\Omega + \Omega_A)^2}{|N + N_A|^2} + \frac{(\Omega - \Omega_A)^2}{|N - N_A|^2}\right]|\phi|^2 rdr$$
$$+ \int\frac{\rho_0}{|N|^2}\left[N_A^2 + \Phi - \Psi_A - 4(\Omega^2 + \Omega_A^2) - \frac{2mEDN}{r}\right.$$
$$\left.\left. - \frac{E(DN)^2}{4}\left(1 + \frac{N_A^2}{|N|^2}\right)\right]|\phi|^2 rdr\right\} = 0 \tag{28}$$

Since the first three integrals are all positive definite, no solution corresponding to $\omega_i \neq 0$ exists if the sum of the quantities in the last integral in Eq. (28) is positive. Therefore we can conclude that the flow is stable if

$$\Phi - \Psi_A - 4(\Omega^2 + \Omega_A^2) + (kW_A + m\Omega_A)^2$$
$$- \frac{r^2(kDW + mD\Omega)}{m^2 + k^2r^2}\left\{\frac{2m\Omega}{r} + \frac{kDW + mD\Omega}{4}\left[1 + \frac{(kW_A + m\Omega_A)^2}{|kW + m\Omega - \omega|^2}\right]\right\} \geqslant 0 \quad (29)$$

anywhere within the flow domain. Equation (29) can be rearranged to

$$\frac{(1 + s^2)\{\Phi - \Psi_A - 4\Omega_A^2 + N_A^2\} - 4\Omega^2}{[DW + s(rD\Omega + 4\Omega)]^2 + \left[(DW + srD\Omega)\dfrac{N_A}{|N|}\right]^2} \geqslant \frac{1}{4} \quad (30)$$

where $s = m/kr$. It can be seen immediately that Eq. (30) will be violated when

$$\Phi - \Psi_A - 4\Omega_A^2 + N_A^2 \leqslant 4\Omega^2/(1 + s^2). \quad (31)$$

Equation (30) is reminiscent of the stability criterion obtained by Fung & Kurzweg (1975) in their study on the stability of swirling flows with radius-dependent density. Like the role played by the Rayleigh-Synge criterion (a condition for centrifugal stability) in their stability criterion, Eq. (31) can be viewed as a precondition to the sufficiency criterion in Eq. (30). However, this sufficiency criterion for zero magnetic forces does not reduce to the one obtained by Fung & Kurzweg (1975). This is a well-known paradox stemming from the fact that the lines of magnetic flux for a fluid with zero resistivity are permanently attached to the fluid (Chandrasekhar 1961). The attachment is contained in the second and the third integral of Eq. (28). If we assume

$$N_A = kW_A + m\Omega_A = 0 \quad (32)$$

throughout the flow domain, Eq. (28) can be written as

$$\omega_i\left\{\int \rho_0(E|D^*\phi|^2 + |\phi|^2)\, rdr\right.$$
$$\left. + \int \frac{\rho_0}{|N|^2}\left[I_2 + I_3 + \Phi - \Psi_A - 4(\Omega^2 + \Omega_A^2) - \frac{2m\Omega EDN}{r} - \frac{E(DN)^2}{4}\right]|\phi|^2 rdr\right\} = 0. \quad (33)$$

The positive definite integrands

$$I_2 = \frac{4m^2\Omega_A^2}{m^2 + k^2r^2} \quad (34)$$

and

$$I_3 = \frac{4k^2r^2}{m^2 + k^2r^2} (\Omega^2 + \Omega_A^2) \tag{35}$$

which were originally not admitted in the sufficiency condition obtained from the second and third integral in Eq. (28), are now recovered. The resultant condition for stability now reads

$$\frac{(1 + s^2)(\Phi - \Psi_A)}{[DW + s(rD\Omega + 4\Omega)]^2} \geqslant \frac{1}{4}. \tag{36}$$

Note that Eq. (36) is valid only with the constraint described by Eq. (32). It is remarkable that even though the constraint involves the magnetic forces in both the axial and azimuthal direction, the sufficiency condition described in Eq. (36) does not depend on the axial magnetic forces. Furthermore, if we compare the numerator in Eq. (30) with that in Eq. (36), a destabilizing term $-4[\Omega^2 + (1 + s^2)\Omega_A^2]$ appearing in the former disappears in the latter as N_A approaches zero. Though this paradox suggests a destabilization of the flow by the magnetic forces, one should keep in mind that both the sufficiency conditions in Eqs. (30) and (36) only represent a bound on stability and are by no means to give the final stability conditions.

Since the sufficiency condition for stability obtained in Eq. (29) or (30) contains the unknown frequency ω, a more general view of flow characteristics in terms of given flow profiles is therefore difficult to see. In the following we will further examine the flow characteristics using other methods and compare the results with the derived sufficiency conditions for some particular cases.

(2) Comparisons with Different Modes

Three different types of perturbations are to be examined and compared with the earlier obtained sufficiency condition.

A. Axisymmetric modes ($m = 0$)

For simplicity, we will neglect the axial magnetic flux, i.e., $W_A = 0$ throughout the flow domain. Equation (20) under the present conditions reads

$$D\,[\rho_0(W - c_k)^2 D^*\psi] - \rho_0\,[k^2(W - c_k)^2 - \Phi + \Psi_A]\psi = 0 \tag{37}$$

where

$$c_k = \frac{\omega}{k} = c_{kr} + ic_{ki}$$

is the axial phase velocity. The constraint in Eq. (32) is now satisfied and the loss of the stabilizing effect by the presence of the axial magnetic flux is recovered. The sufficient condition for stability reduced from Eq. (36) becomes

$$J_k \geqslant \frac{1}{4} \tag{38}$$

where

$$J_k = \frac{\Phi - \Psi_A}{(DW)^2} \tag{39}$$

which is reminiscent of the Richardson number in two-dimensional stratified flows. Condition (38) was first derived by Howard & Gupta (1962) for homogeneous fluids. In the absence of axial flows, Eq. (37) together with the boundary conditions forms a Sturm Liouville system for which

$$\Phi - \Psi_A \geqslant 0 \tag{40}$$

is a necessary and sufficient condition for stability. The above condition was first derived by Michael (1954) for homogeneous fluids and will be called the generalized Michael condition. An alternative way to examine the influence of flow quantities on stability can be obtained as follows.

Multiplying Eq. (37) by $r\bar{\psi}$ and integrating the resulting equation throughout the flow domain, we obtain

$$\int (W - c_k)^2 \, X_k dr - \int \rho_0(\Phi - \Psi_A) \, |\psi|^2 r dr = 0 \tag{41}$$

where

$$X_k = \rho_0 \, (|D^*\psi|^2 + k^2|\psi|^2)r \geqslant 0.$$

Solving Eq. (38) for c_k results in

$$c_k = \frac{\int WX_k dr \pm \sqrt{\int X_k dr \int \rho_0 \, (\Phi - \Psi_A) \, |\psi|^2 r dr - \delta_k}}{\int X_k dr} \tag{42}$$

where

$$\delta_k = \int X_k dr \int W^2 X_k dr - (\int WX_k dr)^2 \geqslant 0$$

for all values of W resulting from the Schwarz inequality. It follows from Eq. (42) that instability of the flow is expected when

$$-\delta_k + \int X_k dr \int \rho_0 (\Phi - \Psi_A)\ |\psi|^2 r dr < 0. \tag{43}$$

It is obvious from the above equation that the axial velocity always destabilizes the flow except for constant W where $\delta_k \equiv 0$. The second term in Eq. (43) represents the contribution from the Rayleigh-Synge and the Alfvén discriminants. Instability will occur when the generalized Michael condition is violated. It is interesting to note that even though both Eqs. (38) and (43) involve similar arguments on the axial velocity and on the generalized Michael condition, they represent different bounds on flow stability. It is obvious from Eq. (43) that violating the generalized Michael condition automatically leads to instability of the flow. However, such a conclusion can not be drawn directly from Eq. (38) since violating the sufficiency condition does not necessary lead to instability.

B. Azimuthal modes ($k = 0$)

The sufficiency condition for $k = 0$ reduced from Eq. (29) reads

$$\frac{J_m}{1 + \Omega_A^2/(\Omega - c_m)^2} \geqslant \frac{1}{4} \tag{44}$$

$$J_m = \frac{r\Omega^2 D\rho_0 - rD(\rho_0\Omega_A^2) + (m^2 - 4)\rho_0\Omega_A^2}{\rho_0(rD\Omega)^2} \tag{45}$$

where

$$c_m = \frac{\omega}{m} = c_{mr} + ic_{mi}$$

and is the angular phase velocity. An analogy can be drawn between the two sufficiency conditions in Eqs. (38) and (44) except in the latter the shear effect is produced by the angular velocity rather than the axial velocity. Equation (44) suggests that the angular velocity plays a dual role in flow stability: the angular velocity itself stabilizes the flow while its gradient destabilizes the flow. This characteristic can further be observed in the following analysis. The influence of the azimuthal magnetic field on flow stability is difficult to see because the sufficiency condition involves the complex phase velocity. An alternative way to examine the flow characteristics is given as follows.

Equation (20) for $k = 0$ is written as

$$D[\rho_0 r^2(N^2 - m^2\Omega_A^2)D^*\psi]$$
$$- \{2mrD[\rho_0(N\Omega - m\Omega_A^2)] + \rho_0 m^2(N^2 - \Phi + 4\Omega^2 + \Psi_A - m^2\Omega_A^2)\}\psi = 0. \qquad (46)$$

Here

$$N = m\Omega - \omega.$$

Multiplying Eq. (46) by $r\bar{\psi}$ and integrating the resultant equation yield

$$\int (N^2 - m^2\Omega_A^2)\,X_m dr + \int \{2mrND(\rho_0\Omega) - m^2[r\Omega^2(D\rho_0) + rD(\rho_0\Omega_A^2)]\}r\,|\psi|^2 dr = 0 \qquad (47)$$

where

$$X_m = \rho_0(r^2\,|D^*\psi|^2 + m^2\,|\psi|^2)r \geqslant 0. \qquad (48)$$

Further we utilize the transform

$$Y_m = \rho_0\,[r^2|D\psi|^2 + (m^2 - 1)\,|\psi|^2]r \geqslant 0 \qquad (49)$$

Eq. (47) can be reduced to a simple quadratic form of the complex azimuthal phase velocity c_m

$$c_m^2\int X_m dr - 2c_m\int \Omega Y_m dr + \int (\Omega^2 - \Omega_A^2)Y_m dr = 0. \qquad (50)$$

Solving (50) for c_m leads us to a conclusion that the flow will be unstable (corresponding to $c_i \neq 0$) when

$$-\delta_m + \int (D\rho_0)r^2\,|\psi|^2 dr \int \Omega^2 Y_m dr + \int X_m dr \int \Omega_A^2 Y_m dr < 0 \qquad (51)$$

where

$$\delta_m = \int Y_m dr \int \Omega^2 Y_m dr - (\int \Omega Y_m dr)^2 \geqslant 0$$

for all values of Ω resulting from the Schwarz inequality. The first term of Eq. (51) represents the shear effect produced by the angular velocity gradient. It is obvious that this term always destabilizes the flow except for uniform rotation where $\delta_m = 0$. Instability in this case is of centrifugal origin because of the absence of shear layers within the flow domain. The second term (51) represents the density variation within the centrifugal force field produced by the rotation of fluid. Positive density gradients stabilize the flow. In the absence of magnetic fields, instability automatically occurs when the density decreases radially outwards regardless of the detailed profile of the angular velocity. Even though it is also suggested by Eq. (44) that negative density gradients may lead to the violation of the sufficient condition for stability, instability, however, can not be concluded. Violating Eq. (44) does not necessarily lead to instability. The last term in Eq. (51) represents the magnetic influence on flow stability. The presence of the azimuthal magnetic force, regardless of its detailed distribution, always stabilizes the azimuthal disturbances. This argument, which cannot be seen clearly from criterion (44), will be supported by an analytical solution to the governing stability equation. Another conclusion we can draw from Eq. (51) is that for uniform rotation flow stability can be guaranteed if the density increases radially outwards. This characteristic is also independent of the profile of the magnetic field and cannot be observed from Eq. (44).

C. Arbitrary modes

Because of the complexity of the arbitrary perturbations, we will ignore the influence from the magnetic field and concentrate on the contribution from the density and velocities of the flow. Under these conditions, the constraint in Eq. (32) is satisfied and, without losing the the stabilizing effect, the sufficient condition for stability reduced from Eq. (36) reads

$$\frac{(1+s^2)\Phi}{[DW + s(rD\Omega + 4\Omega)]^2} \geqslant \frac{1}{4}. \tag{52}$$

This condition was first derived by Fung & Kurzweg (1975) in their study on heterogeneous swirling flows. Readers are referred to their detailed discussions on the condition. Reminiscent of the statically stable condition for density encountered in two-dimensional shear flows, the Rayleigh-Synge criterion ($\Phi \geqslant 0$) acts as a condition for centrifugal stabilities.

In the following, we will further investigate the stability characteristics by adopting the method used in the axisymmetric and the azimuthal cases. The integral equation thus obtained is

$$\int N^2 X dr - 2\int m\Omega N \frac{\rho_0 D(r^2|\psi|^2)}{m^2 + k^2r^2} dr - \int \left[\Phi - \frac{4m^2\Omega^2}{m^2+k^2r^2}\right] \rho_0 |\psi|^2 r dr = 0 \tag{53}$$

where

$$X = \rho_0 \left[\frac{r^2}{m^2+k^2r^2} |D^*\psi|^2 + |\psi|^2\right] r \geqslant 0. \tag{54}$$

We further utilize

$$\mathrm{Y} = \frac{\rho_0 [r^2 |D\psi|^2 + (k^2 r^2 + m^2 - 1)|\psi|^2] r}{m^2 + k^2 r^2} \geqslant 0. \tag{55}$$

Equation (53) is transformed to a simple quadratic form of ω

$$\omega^2 \int \mathrm{X} dr - 2\omega \int (kW\mathrm{X} + m\Omega \mathrm{Y}) dr + \int (k^2 W^2 \mathrm{X} + 2kWm\Omega \mathrm{Y} + m^2 \Omega^2 \mathrm{Y}) dr$$
$$- \int \rho_0 \left[\Phi + \frac{2m^2 \Omega^2}{m^2 + k^2 r^2} \right] \frac{k^2 r^2}{m^2 + k^2 r^2} |\psi|^2 r dr = 0. \tag{56}$$

Instabilities are expected whenever

$$-\delta_W - \delta_\Omega - \delta_s - \delta_f + \int (D\rho_0) \frac{r^2 |\psi|^2}{m^2 + k^2 r^2} dr \int m^2 \Omega^2 \mathrm{Y} dr$$
$$+ \int \mathrm{X} dr \int \rho_0 \Phi \frac{k^2 r^2}{m^2 + k^2 r^2} |\psi|^2 r dr < 0. \tag{57}$$

Here

$$\delta_W = \int \mathrm{X} dr \int k^2 W^2 \mathrm{X} dr - \left(\int kW\mathrm{X} dr\right)^2$$
$$\delta_\Omega = \int \mathrm{Y} dr \int m^2 \Omega^2 \mathrm{Y} dr - \left(\int m\Omega \mathrm{Y} dr\right)^2$$
$$\delta_s = 2\left(\int \mathrm{X} dr \int kWm\Omega \mathrm{Y} dr - \int kW\mathrm{X} dr \int m\Omega \mathrm{Y} dr\right)$$
$$\delta_f = 2\left[\int m^2 \Omega^2 \mathrm{Y} dr \int \frac{k^2 r^2}{(m^2 + k^2 r^2)^2} \rho_0 |\psi|^2 r dr - \int \mathrm{X} dr \int \frac{m^2 \Omega^2 k^2 r^2}{(m^2 + k^2 r^2)^2} \rho_0 |\psi|^2 r dr\right].$$

Since X and Y are positive definite, it follows from the Schwarz inequality that both δ_W and δ_Ω are positive definite for all values of W and Ω, representing the shear effects conveyed by the gradients of the axial and tangential velocities. As also implied by condition (52) they both destabilize the flow except for constant axial and angular velocities where $\delta_W = \delta_\Omega = 0$. The third and fourth terms are respectively the influence of the perturbation directions on the shear effect produced by the velocities and on the centrifugal force created by the fluid rotation. Both can be either positive or negative, implying that they can either stabilize or destabilize the flow. This stabilization or destabilization will not be seen until solutions to the governing stability equation are obtained. The fifth term in (57) is the effect of density variations in the centrifugal force field. Positive density gradients stabilize the flow. The last term in (57) involves the Rayleigh-Synge discriminant and stabilizes the flow if the Rayleigh-Synge criterion ($\Phi \geqslant 0$) is satisfied. Condition (57) can reduce to condition (43) for the axisymmetric case and to condition (51) for the azimuthal case if the magnetic fields are deleted.

(3) Bounds on Unstable Waves

For instability, a bound on the growth rate can readily be obtained from Eq. (28). Since the first three integrals in the equation are all positive definite, it follows, as an opposite to the sufficiency condition for stability in Eq. (29), that the integrand in the last integral must be negative somewhere within the flow domain. This leads to a bound on the growth rate such that

$$\omega_i^2 < \left\{ \frac{N_A^2 (DW + srD\Omega)^2}{4[(1+s^2)(\Phi - \Psi_A - 4\Omega_A^2 + N_A^2) - 4\Omega^2] - [DW + s(rD\Omega + 4\Omega)]^2} \right\}_{\max} . \tag{58}$$

The above bound will be meaningless for those profiles satisfying $N_A = 0$. A bound for those cases can also be obtained from Eq. (33) and the growth rate is then bounded by

$$\omega_i^2 < \left\{ \frac{[DW + s(rD\Omega + 4\Omega)]^2}{4(1+s^2)} - (\Phi - \Psi_A) \right\}_{\max} . \tag{59}$$

In the following we will further investigate upper bounds for possible unstable waves and compare them with the stability characteristics previously obtained. Three special cases will be discussed.

A. The axisymmetric case ($m = 0$)

For mathematical simplicity in this case, we further ignore the axial magnetic field, i.e., $W_A = 0$. The integral equation for the present case, reduced from Eq. (28), becomes

$$\int \rho_0 (|D^*\phi|^2 + k^2|\phi|^2) r dr + \int \rho_0 k^2 \left[\Phi - \Psi_A - \frac{(DW)^2}{4} \right] \frac{|\phi|^2}{|N|^2} r dr = 0 \tag{60}$$

as the governing relation for possible instabilities. From Eqs. (20), (25) and (60) we follow the procedures used by Kochar & Jain (1979) in their derivation of the semi-ellipse theorem in two-dimensional stratified flows to obtain

$$\left\{ c_{kr} - \frac{W_{\max} + W_{\min}}{2} \right\}^2 + \left\{ 1 + \frac{4(J_k)_{\min}}{[1 + \sqrt{1 - 4(J_k)_{\min}}]^2} \right\} c_{ki}^2 \leq \left\{ \frac{W_{\max} - W_{\min}}{2} \right\}^2 \tag{61}$$

where J_k as given in Eq. (39) is restricted to be less than 1/4 if Eq. (61) is valid. The subscripts max and min represent the maximum and minimum of the quantities within the flow domain. The semi-ellipse theorem for the axisymmetric case thus states that the unstable axial phase velocity must lie within a semi-ellipse in the complex phase velocity plane as described by Eq. (61). The semi-ellipse is bounded by the upper and lower limits of the axial flow velocity and is exactly the same the one in two-dimensional stratified flow except that the local Richardson number is defined as in Eq. (39) instead.

B. The azimuthal case ($k = 0$)

By using the Boussinesq approximation, Acheson (1973) was able to demonstrate the westward drift phenomenon in his study on hydromagnetic wavelike instabilities in a rapidly rotating stratified fluid. In addition, a quadrant theorem reminiscent of the semi-circle theorem in two-dimensional stratified flows was also derived for slow amplifying waves. Even though his criteria were obtained for nonaxisymmetric modes under certain assumptions, any attempt to infer that the azimuthal modes carry the same characteristics as the nonaxiymmetric ones is, at best, uncertain. In the present case, we will investigate this uncertainty. The procedure to be used to construct the bound for instabilities is similar to the one used by Fung (1982) in his study of nonaxisymmetric instability on rotating flows except that the present case is complicated by the magnetic field.

From the derivation of the sufficiency condition for stability, the integral equation of unstable waves for $k = 0$ reduced from Eq. (28) is written as

$$\int \rho_0(r^2|D^*\phi|^2 + m^2|\phi|^2)\,rdr + \int \frac{\rho_0 m^2 r^2 \Omega_A^2}{|N|^2}\left\{D|\phi| - \left[\frac{1}{r} + \frac{mrD\Omega\,(m\Omega - \omega_r)}{|N|^2}\right]|\phi|\right\}^2 rdr + \int \frac{\rho_0 m^2}{|N|^2}\left[J_m - \frac{1}{4}\left(1 + \frac{m^2\Omega_A^2}{|N|^2}\right)\right] r^2(D\Omega)^2|\phi|^2 rdr = 0. \tag{62}$$

Substituting Eq. (25) into (62) and applying the Schwarz inequality lead to the inequality

$$\int X_m dr \leq \frac{1}{4c_{mi}^2}\left[\sqrt{1 + \frac{(\Omega_A)^2_{\max}}{c_{mi}^2} - 4(J_m)_{\min}} + 1\right]^2 \int r^2(D\Omega)^2\rho_0|\psi|^2 rdr. \tag{63}$$

Next we separate the real and imaginary parts of Eq. (50) for instability into

$$\int (\Omega^2 - 2c_{mr}\Omega - \Omega_A^2)\,Y_m dr + (c_{mr}^2 - c_{mi}^2)\int X_m dr = 0 \tag{64}$$

and

$$\int \Omega Y_m dr - c_{mr} \int X_m dr = 0. \tag{65}$$

Let Ω_{max} be the upper bound and Ω_{min} be the lower bound of the angular velocity within the flow field. Because

$$\int (\Omega - \Omega_{max})(\Omega - \Omega_{min}) Y_m dr \leqslant 0 \tag{66}$$

incorporating Eqs. (64) and (65) into (66) yields

$$\begin{aligned}\{c_{mr}^2 + c_{mi}^2 - (\Omega_{max} + \Omega_{min})\, c_{mr} + \Omega_{max}\Omega_{min} + (\Omega_A)^2_{min}\} \int X_m dr \\ + \int [\Omega_{max}\Omega_{min}(D\rho_0) + D(\rho_0\Omega_A^2)] r^2 |\psi|^2 dr \leqslant 0.\end{aligned} \tag{67}$$

Equation (67) implies that the complex angular phase velocity will no longer be bounded by a semicircle if $\Omega_{max}\Omega_{min}D\rho_0 + D(\rho_0\Omega_A^2) < 0$. This characteristic will be demonstrated by an analytical solution to the stability equation to be given in the next section. To construct a bound for possible unstable waves, we assume that

$$\Omega_{max}\Omega_{min}D\rho_0 + D(\rho_0\Omega_A^2) \geqslant 0 \tag{68}$$

and combine Eq. (63) and (67) to obtain

$$\left\{c_{mr} - \frac{\Omega_{max} + \Omega_{min}}{2}\right\}^2 + \left\{1 + \frac{4\left[\dfrac{r\Omega_{max}\Omega_{min}D\rho_0 + rD(\rho_0\Omega_A^2)}{\rho_0(rD\Omega)^2}\right]_{min}}{\left[\sqrt{1 + \dfrac{(\Omega_A^2)_{max}}{c_{mi}^2} - 4(J_m)_{min}} + 1\right]^2}\right\} c_{mi}^2$$

$$\leqslant \left\{\frac{\Omega_{max} - \Omega_{min}}{2}\right\}^2 - (\Omega_A^2)_{min}. \tag{69}$$

Thus the complex angular phase velocity for unstable waves will be bounded by a curve described in Eq. (69). Another conclusion we can directly draw from Eq. (69) is that stabilities will occur if the minimum absolute value of the angular Alfvén velocity exceeds half of the maximum angular velocity difference, i.e.,

$$|\Omega_A|_{min} \geqslant \frac{1}{2}(\Omega_{max} - \Omega_{min}). \tag{70}$$

One should keep in mind that both criteria, Eq. (69) and Eq. (70), are valid only with the restriction given in Eq. (68). The bound described by Eq. (69) depends not only on the density but also on the upper and lower bounds of the angular velocity and of the Alfvén waves. The bound in Eq. (69) reduces to the semi-ellipse theorem derived by Fung (1982) should the azimuthal magnetic field be deleted from the flow. The bound in Eq. (69) also indicates that the amplifying waves do not necessarily propagate against the basic rotation. Accordingly, the westward drift that exists in the arbitrary amplifying modes does not prevail in the azimuthal case.

C. Uniform rotation and constant angular Alfvén wave

A solution for azimuthal modes described by a semi-circle can be derived for all unstable waves of the flow with uniform rotation and constant angular Alfvén velocity regardless of the particular form of the density distribution. For $\Omega = \Omega_0$ and $\Omega_A = \Omega_{A0}$, Eq. (46) under the present assumption yields a simple form

$$D(\rho_0 r^2 D^* u) - [\Lambda r (D\rho_0) + m^2 \rho_0] u = 0 \tag{71}$$

where

$$\Lambda = \Lambda_r + i\Lambda_i = 1 - \frac{c_m^2}{(\Omega_0 - c_m)^2 - \Omega_{A0}^2} \tag{72}$$

and

$$\Lambda_r = \frac{1}{|(\Omega_0 - c_m)^2 - \Omega_{A0}^2|^2} \{[(\Omega_0 - c_{mr})^2 - \Omega_{A0}^2 - c_{mr}^2]\,[(\Omega_0 - c_{mr})^2 - \Omega_{A0}^2 - c_{mi}^2]$$
$$+ 4\Omega_0 (\Omega_0 - c_{mr})\, c_{mi}^2\}$$

$$\Lambda_i = \frac{2\Omega_0 c_{mi}}{|(\Omega_0 - c_m)^2 - \Omega_{A0}^2|^2} \left\{c_{mr}^2 - \Omega_0 \left[1 - \frac{\Omega_{A0}^2}{\Omega_0^2}\right] c_{mr} + c_{mi}^2\right\}.$$

Equation (71) and the conditions that u vanishes at the inner and outer boundaries form a Sturm-Louiville system having the following characteristics: (1) Λ is always real indicating $\Lambda_i = 0$, and (2) Λ and $D\rho_0$ are of opposite signs. These characteristic can also be shown by applying the integral method to Eq. (71) or directly obtained from Eq. (50) under the present assumptions. The first characteristic states that the complex angular phase velocity for all unstable waves must lie on a semi-circle described by

$$\left\{c_{mr} - \frac{\Omega_0}{2}\left[1 - \frac{\Omega_{A0}^2}{\Omega_0^2}\right]\right\}^2 + c_{mi}^2 = \left\{\frac{\Omega_0}{2}\left[1 - \frac{\Omega_{A0}^2}{\Omega_0^2}\right]\right\}^2 \tag{73}$$

and since

$$\Lambda = \Lambda_r = \frac{(\Omega_0^2 - \Omega_{A0}^2)}{|(\Omega_0 - c_{mr})^2 - \Omega_{A0}^2|^2}\left[1 - \frac{c_{mr}}{\Omega_0}\right] \geqslant 0 \tag{74}$$

must hold for all unstable waves, the second characterisitic clearly demonstrates that instabilities are impossible when $D\rho_0 \geqslant 0$. As a matter of fact this conclusion can also be drawn directly from Eq. (51) even for arbitrary angular Alfvén velocity. Instabilities of this type are certainly of centrifugal origin since no shear layers exist within the flow field. In the absence of the azimuthal magnetic force, the present semi-circle for unstable waves reduces to the one obtained by Fung (1982) in his analysis on nonaxisymmetric instability of vortex flows. Once again, the unstable waves do not propagate against the basic rotation unless $\Omega_{A0}^2 > \Omega_0^2$.

SOME EXACT SOLUTIONS

To demonstrate the validity of the sufficiency condition for stability and of the bounds for unstable waves, we proceed to construct several flow profiles and obtain exact solutions to the governing stability equation. Three types of vortex flows will be considered.

(1) Uniform Rotation

First consider a flow profile with the distribution as follows:

$$\begin{array}{ll} \Omega(r) = \Omega_2 & \Omega_A(r) = \Omega_{A2} \\ W(r) = W_2 & W_A(r) = W_{A2}. \\ \rho_0(r) = \rho_2(r/R)^\sigma & \end{array} \tag{75}$$

Here R, σ and all the quantities with the subscript 2 are arbitrary constants. The perturbation velocity in the radial direction obtained by solving Eq. (20) is

$$\begin{aligned} u_2 = N_2 r^{-\frac{\sigma}{2}-1} \Bigg\{ & A\left[\frac{\sigma}{2} + \frac{2m(N_2\Omega_2 - N_{A2}\Omega_{A2})}{N_2^2 - N_{A2}^2} + \frac{kgrI_\nu'(kgr)}{I_\nu(kgr)}\right] I_\nu(kgr) \\ & + B\left[\frac{\sigma}{2} + \frac{2m(N_2\Omega_2 - N_{A2}\Omega_{A2})}{N_2^2 - N_{A2}^2} + \frac{kgrK_\nu'(kgr)}{K_\nu(kgr)}\right] K_\nu(kgr)\Bigg\} \end{aligned} \tag{76}$$

with the corresponding perturbation pressure equal to

$$q_2 = -i(N_2^2 - N_{A2}^2)g^2\rho_2 \left(\frac{r}{R}\right)^{\sigma} r^{-\frac{\sigma}{2}} [AI_\nu(kgr) + BK_\nu(kgr)] \tag{77}$$

where

$$q = 1 - \frac{\sigma(\Omega_2^2 - \Omega_{A2}^2)}{N_2^2 - N_{A2}^2} - 4\left(\frac{N_2\Omega_2 - N_{A2}\Omega_{A2}}{N_2^2 - N_{A2}^2}\right)^2$$

$$\nu = \left\{m^2g^2 + \left[\frac{\sigma}{2} + \frac{2m(N_2\Omega_2 - N_{A2}\Omega_{A2})}{N_2^2 - N_{A2}^2}\right]^2\right\}^{1/2}$$

$$N_2 = kW_2 + m\Omega_2 - \omega$$

$$N_{A2} = kW_{A2} + m\Omega_{A2}$$

and $I_\nu(kgr)$ and $K_\nu(kgr)$ are the modified Bessel functions of the first and second kind of order ν. The prime denotes the total derivative with respect to the arguments shown. We further consider the present flow to be confined within two solid boundaries located at R_1 and R_2. The secular relation obtained by applying the boundary conditions $u_2(R_1) = u_2(R_2) = 0$ is

$$\begin{vmatrix} \left[\frac{\sigma}{2} + \frac{2m(N_2\Omega_2 - N_{A2}\Omega_{A2})}{N_2^2 - N_{A2}^2} + \frac{\kappa_1 g I_\nu'(\kappa_1 g)}{I_\nu(\kappa_1 g)}\right] I_\nu(\kappa_1 g) & \left[\frac{\sigma}{2} + \frac{2m(N_2\Omega_2 - N_{A2}\Omega_{A2})}{N_2^2 - N_{A2}^2} + \frac{\kappa_1 g K_\nu'(\kappa_1 g)}{K_\nu(\kappa_1 g)}\right] K_\nu(\kappa_1 g) \\ \left[\frac{\sigma}{2} + \frac{2m(N_2\Omega_2 - N_{A2}\Omega_{A2})}{N_2^2 - N_{A2}^2} + \frac{\kappa_2 g I_\nu'(\kappa_2 g)}{I_\nu(\kappa_2 g)}\right] I_\nu(\kappa_2 g) & \left[\frac{\sigma}{2} + \frac{2m(N_2\Omega_2 - N_{A2}\Omega_{A2})}{N_2^2 - N_{A2}^2} + \frac{\kappa_2 g K_\nu'(\kappa_2 g)}{K_\nu(\kappa_2 g)}\right] K_\nu(k_2 g) \end{vmatrix} = 0 \tag{78}$$

where

$$\kappa_j = kR_j \qquad j = 1, 2.$$

It is difficult to observe the general behavior of Eq. (78) because both the argument and the order of the modified Bessel functions involve the complex eigenfrequency. However, stability behaviors for some special cases can still be observed and compared with the earlier obtained criteria. One such case is the axisymmetric mode with zero axial magnetic field. The asymptotic expansion of the Bessel functions for large axial wave numbers allows us to obtain the explicit solution

$$c_k = \frac{\omega}{k} = W_2 \pm \left[\frac{(\sigma + 4)\Omega_2^2 - \sigma\Omega_{A2}^2}{k^2 + a^2\pi^2/(R_2 - R_1)^2}\right]^{1/2} \tag{79}$$

where

$$a = 0, \ \pm 1, \ \pm 2, \ .....$$

Equation (79) shows the flow will be stable if and only if

$$(\sigma + 4)\Omega_2^2 - \sigma\Omega_{A2}^2 \geqslant 0. \tag{80}$$

This is exactly what the generalized Michael condition predicts in Eq. (40). The stability domain is plotted in Fig. 1. For axisymmetric modes, the presence of the azimuthal magnetic field stabilizes the flow for negative density gradients and destabilizes the flow for positive density gradients.

Another special case is the one for azimuthal modes. The solution to Eq. (78) for $k = 0$ is found to be

$$c_m = \frac{\omega}{m} = \frac{(\sigma + h)\Omega_2 \pm \sqrt{(\sigma + h)\ (\sigma\Omega_0^2 + h\Omega_{A2}^2)}}{h} \tag{81}$$

where

$$h = m^2 + \frac{\sigma^2}{4} + \left[\frac{a\pi}{\ln(R_1/R_2)}\right]^2$$

and a is an arbitrary integer. Because $(\sigma + h)$ is always positive definite, the flow will be stable if

$$\sigma\Omega_0^2 + \left\{m^2 + \frac{\sigma^2}{4} + \left[\frac{a\pi}{\ln(R_1/R_2)}\right]^2\right\}\Omega_A^2 \geqslant 0. \tag{82}$$

Unlike the stability behavior in the axisymmetric case the angular Alfvén velocity always stabilizes the azimuthal disturbances. This conclusion is consistent with the stability characteristics predicted by Eq. (51) for azimuthal modes. The stability boundaries described in Eq. (82) are plotted in Fig. 2. Due to the presence of the azimuthal magnetic field, the flow can still be stable even when the density is decreasing radially outwards. Another observable stability characteristic in the case of azimuthal modes is that the solution for unstable waves in Eq. (81) lies on the semi-circle as prescribed by Eq. (73).

(2) Two-Region Flow

The second flow profile we would like to investigate is a two-region flow with the interface between the inner and outer region located at $r = R$. The flow distributions are as follows:

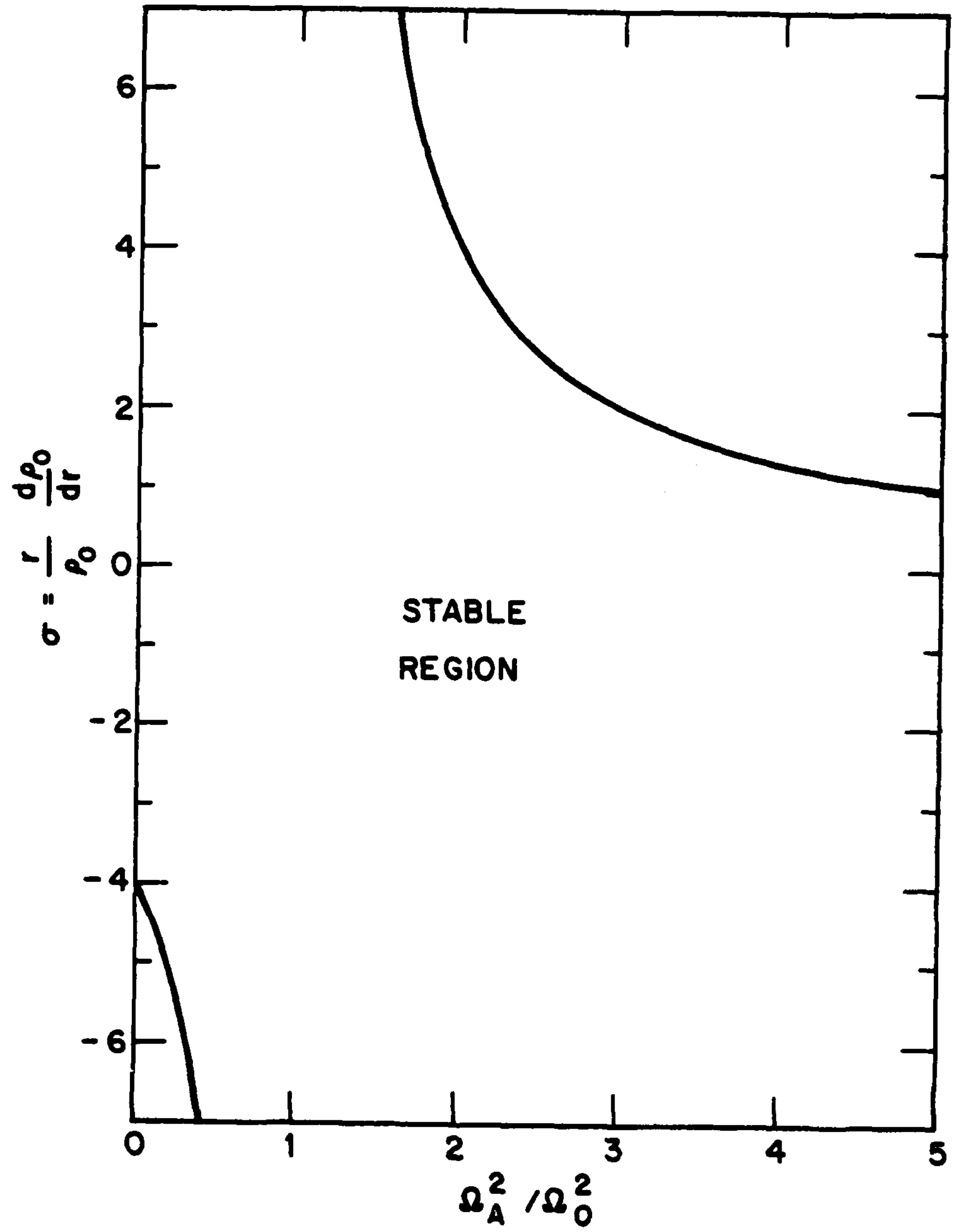

Fig. 1 — The stability domain for axisymmetric modes

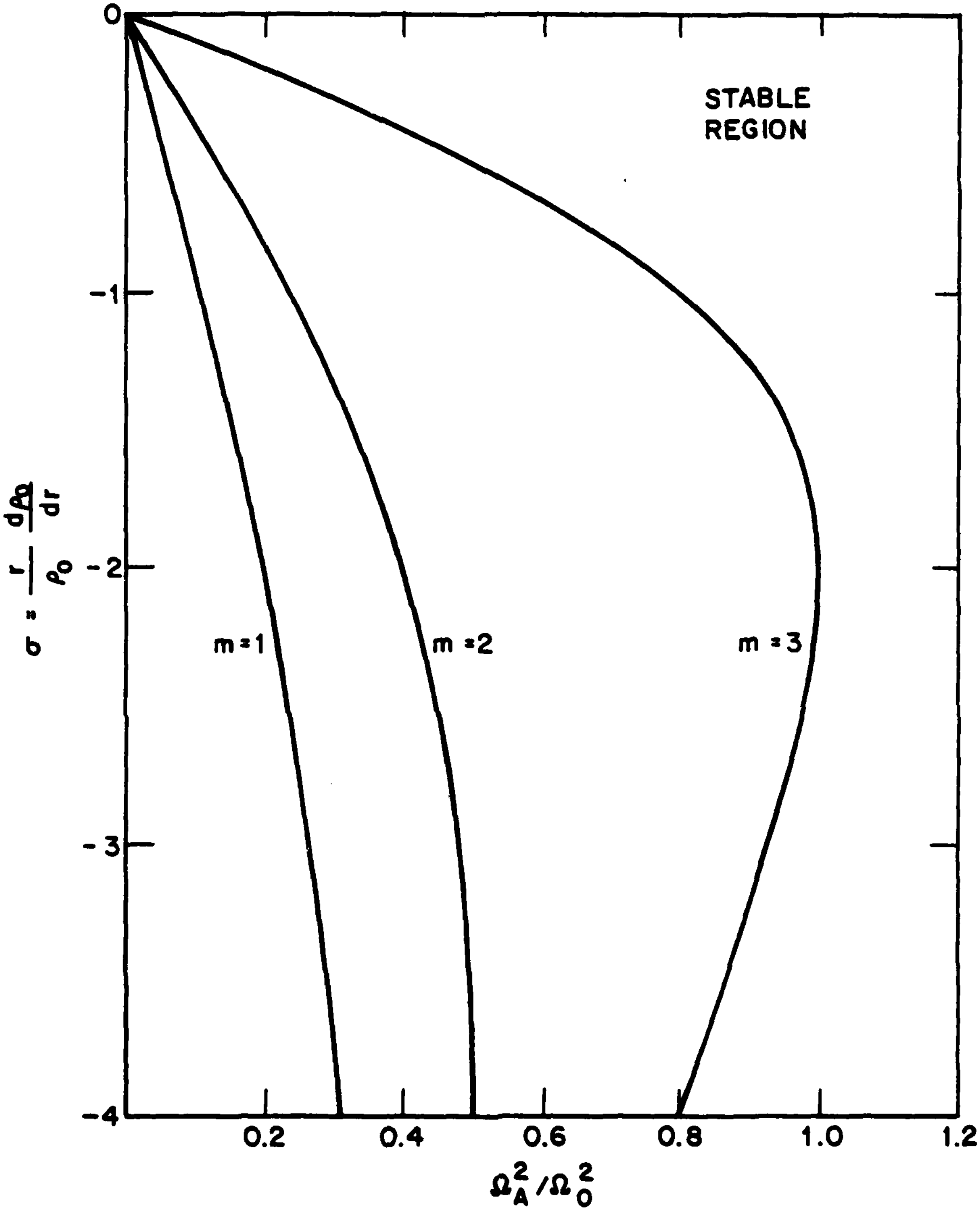

Fig. 2 — The stability for azimuthal modes

$$\Omega(r) = \Omega_j$$
$$W(r) = W_j$$
$$\Omega_A(r) = \Omega_{Aj} \qquad j = 1,\, 2$$
$$W_A(r) = W_A$$
$$\rho_0(r) = \rho_j$$

where all the quantities with index j are constants. After matching the interfacial conditions in Eqs. (23) and (24), the secular relation for stability is found to be:

$$\rho_1 \left\{ \frac{g_1^2 (N_1^2 - N_{A1}^2)}{\dfrac{2m(N_1\Omega_1 - N_{A1}\Omega_{A1})}{N_1^2 - N_{A1}^2} + \dfrac{\kappa g_1 I_m'(\kappa g_1)}{I_m(\kappa g_1)}} + (\Omega_1^2 - \Omega_{A1}^2) \right\}$$
$$- \rho_2 \left\{ \frac{g_2^2 (N_2^2 - N_{A2}^2)}{\dfrac{2m(N_2\Omega_2 - N_{A2}\Omega_{A2})}{N_2^2 - N_{A2}^2} + \dfrac{\kappa g_2 K_m'(\kappa g_2)}{K_m(\kappa g_2)}} + (\Omega_2^2 - \Omega_{A2}^2) \right\} = 0 \tag{83}$$

where

$$g_j = \left[1 - \frac{4(\Omega_j^2 - \Omega_{Aj}^2)}{N_j^2 - N_{Aj}^2} - 4 \left[\frac{N_j \Omega_{Aj} - N_{Aj}\Omega_j}{N_j^2 - N_{Aj}^2} \right] \right]^{\frac{1}{2}}$$
$$N_j = kW_j + m\Omega_j - \omega$$
$$N_{Aj} = kW_{Aj} + m\Omega_{Aj} \qquad j = 1,\, 2$$

and

$$\kappa = kR.$$

If the axial Alfvén velocities are neglected and the perturbations are constrained to be axisymmetric, the asymptotic expansion of the modified Bessel functions for large axial wave numbers allows us to solve Eq. (83) and the resultant condition for stability is

$$\kappa \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right) [\rho_2(\Omega_2^2 - \Omega_{A2}^2) - \rho_1(\Omega_1^2 - \Omega_{A1}^2)] - k^2(W_1 - W_2)^2 \geqslant 0. \tag{84}$$

This condition, composed of the centrifugal force jump and the axial velocity difference at the interface, is consistent with Eq. (43) under the same assumption. The flow will be unstable if the centrifugal force at the interface is decreasing outwards. The axial velocity difference always destabilizes the flow except for $W_1 = W_2$. In that case, the generalized Michael condition in (40) is recovered as

$$\rho_2(\Omega_2^2 - \Omega_{A2}^2) - \rho_1(\Omega_1^2 - \Omega_{A1}^2) \geqslant 0 \tag{85}$$

for stability. As a matter of fact, condition (85) can easily be obtained by integrating Eq. (40) across the interface, since the flow is stable within the inner and the outer regions.

Another solution we would like to obtain from Eq. (83) is one for the azimuthal modes. For $k = 0$, Eq. (83) yields a simple solution

$$c_m = \frac{(m-1)\rho_1\Omega_1 + (m+1)\rho_2\Omega_2 \pm \sqrt{\Theta}}{m(\rho_1+\rho_2)} \tag{86}$$

where

$$\begin{aligned}\Theta = &-(m^2-1)\rho_1\rho_2(\Omega_1-\Omega_2)^2 + (\rho_2-\rho_1)\,[(m-1)\rho_1\Omega_1^2 + (m+1)\rho_2\Omega_2^2] \\ &+ m(\rho_1+\rho_2)\,[(m-1)\rho_1\Omega_{A1}^2 + (m+1)\rho_2\Omega_{A2}^2].\end{aligned} \tag{87}$$

Stability of the flow will be guaranteed if

$$\Theta \geqslant 0. \tag{88}$$

As in our previous discussion of Eq. (51) for azimuthal modes, the first term in Eq. (87) is the shear influence generated by the difference in the angular velocities at the interface which always destabilizes the flow. The second term represents the effect of the density variation in the centrifugal force field created by the rotation of the fluids in the inner and outer regions. Stabilization effect requires the density gradient at the interface to be positive. The last term in (87) is the effect of the azimuthal magnetic field which always stabilizes the flow.

For instability, a semi-circle bound, if valid for the present profile, would read

$$\left\{c_{mr} - \frac{\Omega_1+\Omega_2}{2}\right\}^2 + c_{mi}^2 \leqslant \left\{\frac{\Omega_2-\Omega_1}{2}\right\}^2. \tag{89}$$

Substituting Eq. (86) into Eq. (89), we find that the above inequality will be satisfied if

$$(\rho_2-\rho_1)\Omega_1\Omega_2 + [(m+1)\rho_2\Omega_{A2}^2 + (m-1)\rho_1\Omega_{A1}^2] \geqslant 0. \tag{90}$$

This satisfies the restriction in Eq. (68) for the validity of a semi-circle bound. Equation (90) also indicates that unstable waves will no longer be bounded by a semi-circle if the inequality is violated. Unlike the semi-circle bound on all unstable waves in two dimensional stratified flows, the semi-circle theorem in vortex flows does not in general provide a bound on unstable waves if the restriction in Eq. (68) is violated.

(3) Three-Region Flow

As the last example to demonstrate the criteria previously derived for uniformly rotating flows subject to azimuthal perturbations, we consider a three-region flow with constant angular velocity Ω_0, constant angular Alfvén velocity Ω_{A0}, and a density distribution specified by

$$\rho_0(r) = \begin{cases} \rho_1 & 0 \leqslant r < R_1 \\ \rho_2\left(\dfrac{r}{R_1}\right)^\sigma & R_1 \leqslant r < R_2 \, . \\ \rho_3 & R_2 \leqslant r < \infty \end{cases}$$

Here ρ_1, ρ_2, ρ_3 and σ are constants that characterize the density profile in the three flow regions with their common interfaces at the radial positions R_1 and R_2. The axial components of the velocity and magnetic field in this case can be arbitrary. With the solutions for the velocity and pressure perturbations given in Eqs. (76) and (77), we obtain, after using the matching conditions in (23) and (24), the secular relation for stability as

$$\frac{m\rho_1 + \frac{\sigma}{2}\rho_2 + (\rho_2 - \rho_1)\Lambda - \rho_2\nu}{m\rho_1 + \frac{\sigma}{2}\rho_2 + (\rho_2 - \rho_1)\Lambda + \rho_2\nu} R_1^{2\nu} = \frac{m\rho_3 - \frac{\sigma}{2}\rho_2^* + (\rho_3 - \rho_2^*)\Lambda + \rho_2^*\nu}{m\rho_3 - \frac{\sigma}{2}\rho_2^* + (\rho_3 - \rho_2^*)\Lambda - \rho_2^*\nu} R_2^{2\nu} \tag{91}$$

where Λ is given in Eq. (72), ν is given in Eq. (77), and $\rho_2^* = \rho_2(R_2/R_1)^\sigma$ is the density of the middle region evaluated at $r = R_2$. Two special cases for the present flow profile will be investigated.

The first special cases is for $\sigma = 0$, i.e., the fluid is homogeneous in the middle region. Equation (91) yields

$$\Delta\left(1 - \frac{\rho_2}{\rho_1}\right)\left(1 - \frac{\rho_3}{\rho_2}\right)\Lambda^2 - m(1 + \Delta)\left(1 - \frac{\rho_3}{\rho_1}\right)\Lambda + m^2\left[\left(1 + \frac{\rho_3}{\rho_1}\right) + \Delta\left(\frac{\rho_2}{\rho_1} + \frac{\rho_3}{\rho_2}\right)\right] = 0 \tag{92}$$

where

$$\Delta = \frac{R_2^{2m} - R_1^{2m}}{R_2^{2m} + R_1^{2m}}.$$

There are four general types of density profiles for the flow under consideration. They are: (a) $\rho_3 > \rho_2 > \rho_1$, (b) $\rho_3 < \rho_2 < \rho_1$, (c) $\rho_3 < \rho_2 > \rho_1$, and (d) $\rho_3 > \rho_2 < \rho_1$. Combining Eqs. (72) and (92), we find that instabilities will occur when

$$\Omega_\delta^2|y| + \Omega_{A0}^2(|x| \pm \sqrt{x^2 - |\xi|}) < 0 \text{ for type (a)}$$
$$\Omega_\delta^2|y| - \Omega_{A0}^2(|x| \pm \sqrt{x^2 - |\xi|}) > 0 \text{ for type (b)} \tag{93}$$

and

$$\Omega_\delta^2|y| + \Omega_{A0}^2(|x| - \sqrt{x^2 + |\xi|}) > 0 \text{ for types (c) and (d)}$$

where

$$x = m\left(1 + \frac{1}{\Delta}\right)\left(1 - \frac{\rho_3}{\rho_1}\right)$$
$$y = 2\left(1 - \frac{\rho_2}{\rho_1}\right)\left(1 - \frac{\rho_3}{\rho_2}\right)$$

and

$$\xi = 4m^2\left(1 - \frac{\rho_2}{\rho_1}\right)\left(1 - \frac{\rho_3}{\rho_2}\right)\left[\frac{1}{\Delta}\left(1 + \frac{\rho_3}{\rho_1}\right) + \left(\frac{\rho_2}{\rho_1} + \frac{\rho_3}{\rho_2}\right)\right].$$

Since the sums of the terms inside the radicals in Eq. (93) are always positive; we can immediately conclude from the conditions in Eq. (93) that the presence of the azimuthal magnetic field, as in our previous discussion of Eq. (51), always stabilizes the flow. It is obvious that the instability condition for type (a) can never be satisfied, as also predicted by Eq. (51), since the density is increasing radially outwards. In the absence of the azmuthal magnetic field, the conditions in Eq. (93) reduce to those in Fung (1982) and the flow is always unstable except for the density profile in type (a). It can also be shown that for instability, all the unstable waves for the density profile in types (b), (c) and (d) lie on the semi-circle described by Eq. (73).

The second special case we would like to examine is a continuously varying density distribution. Equation (91) for $\rho_1 = \rho_2$ and $\rho_3 = \rho_2(R_1/R_2)^\sigma$ yields the simple form

$$(2m^2 + \sigma\Lambda - 2m\nu)R_1^{2\nu} = (2m^2 + \sigma\Lambda + 2m\nu)R_2^{2\nu}. \tag{94}$$

Equation (94) has the same form as the one (Eq. 22) in Fung & Kurzweg (1975) except for the terms involving Λ and ν due to the presence of the magnetic field. A simple solution to Eq. (94) for $\nu = 0$ allows us to solve for the complex phase velocity as

$$c_m = \frac{(4m^2 + \sigma^2 + 4\sigma)\Omega_0 \pm \sqrt{(4m^2 + \sigma^2 + 4\sigma)\,[4\sigma\Omega_\delta^2 + (4m^2 + \sigma^2)\,\Omega_{A0}^2]}}{4m^2 + \sigma^2}. \tag{95}$$

Since the sum of the terms in the first bracket inside the radical is positive, we can again conclude the azimuthal Aflvén velocity also stabilizes the flow. Furthermore, the complex phase velocity for unstable waves in this case also lies on a semi-circle as predicted by Eq. (73).

CONCLUSIONS

Some general stability criteria for a general type of vortex flows of conducting fluids with axial velocity components under the influence magnetic field are derived. Exact solutions to the governing stability equation for some special flow profiles are obtained and compared with the earlier derived criteria of a less general nature.

The derived sufficiency condition for stability is generally unseparable from the complex eigenfrequency because of the presence of the axial and the azimuthal magnetic field. To further investigate the roles played by the density, velocity and magnetic field in flow stability, three types of perturbation conditions are investigated and compared with the sufficiency condition. It was shown that densities that increase radially outwards always have a stabilizing effect. Unlike the axial velocity which always destabilizes the flow, the angular velocity plays a dual role in flow stability. While the gradient of the velocity generates shear effects which destabilize the flow, the rotation of the velocity creates a centrifugal force field which stabilizes or destabilizes the flow depending on the sign of the density gradient. If perturbations to the flow are restricted to be azimuthal, the magnetic field, regardless of its detailed distribution, always stabilizes the flow.

Several bounds on unstable waves are also obtained and compared with some exact solutions to the stability equation. For axisymmetric instabilities, a semi-ellipse theorem is proved to be valid in the absence of the axial magnetic field. For azimuthal instabilities, an upper bound on the complex phase velocity, reminiscent of the semi-ellipse theorem in the non-magnetic case, is derived with a restriction. Such a restriction indicates, as supported by an exact solution to the stability equation, that not all unstable waves are bounded by a semi-circle in the complex phase velocity plane. For flows with uniform rotations and constant Alfvén velocities subject to azimuthal perturbations, instabilities can only occur when the density decreases radially outwards. Furthermore, the phase velocity for such instabilities, regardless of the detailed density distribution, must lie on a semi-circle in the complex velocity plane. The semi-ellipse and semi-circle bounds derived for the azimuthal modes clearly show that the amplifying waves do not necessarily propagate against the basic rotation. As a conclusion, therefore, the westward drift derived for arbitrary nonaxisymmetric modes does not generally prevail in the azimuthal case.

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